A GENERALISATION OF THE RIEMANNIAN LINE-ELEMENT*

BY

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1. In a manifold of N dimensions and coördinate system x^i, let $P(x^i)$ and $Q(x^i + dx^i)$ be two points with infinitesimal coördinate differences. Our fundamental postulate is as follows:

POSTULATE. *The points P and Q define an invariant infinitesimal line-element ds, expressible as a function of $x^1, x^2, \cdots, x^N, dx^1, dx^2, \cdots, dx^N$.*

Obviously ds must be homogeneous of the first degree in the differentials, and we write

$$ds^2 = F(x^1, \cdots, x^N; dx^1, \cdots, dx^N), \tag{1.1}$$

where F is homogeneous of the second degree in the differentials.† We shall in general write

$$F(x^1, \cdots, x^N; \xi^1, \cdots, \xi^N) = F(x; \xi). \tag{1.2}$$

The further essential postulate in the differential geometry of Riemann is

$$F(x; dx) = g_{ij}\, dx^i\, dx^j, \tag{1.3}$$

where g_{ij} are functions of the coördinates only. In the present paper I wish to develope the more obvious deductions from (1.1), without assuming (1.3).

2. For a coördinate transformation $x^i = x^i(x'^1, \cdots, x'^N)$, we have, writing $\dot{x}^i = dx^i/dt$,

$$\dot{x}^i = \frac{\partial x^i}{\partial x'^j} \dot{x}'^j, \tag{2.1}$$

and therefore

$$\frac{\partial \dot{x}^i}{\partial \dot{x}'^j} = \frac{\partial x^i}{\partial x'^j}; \tag{2.2}$$

also

$$\frac{d}{dt} \frac{\partial x^i}{\partial x'^j} = \frac{\partial^2 x^i}{\partial x'^k\, \partial x'^j} \dot{x}'^k = \frac{\partial \dot{x}^i}{\partial x'^j}. \tag{2.3}$$

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* Presented to the Society, December 30, 1924.

† Cf. P. Finsler, *Über Kurven und Flächen in allgemeinen Räumen*, Dissertation, Göttingen, 1918, p. 33, and J. A. Schouten, *Der Ricci-Kalkül*, Berlin, 1924, p. 36.

If ψ be any invariant function of the coördinates and their first derivatives with respect to t,

$$\frac{\partial \psi}{\partial \dot{x}'^i} = \frac{\partial \psi}{\partial \dot{x}^j} \frac{\partial \dot{x}^j}{\partial \dot{x}'^i} = \frac{\partial \psi}{\partial \dot{x}^j} \frac{\partial x^j}{\partial x'^i}, \qquad \text{by (2.2).}$$

Thus $\partial \psi / \partial \dot{x}^i$ is a covariant vector. Also

$$\frac{\partial^2 \psi}{\partial \dot{x}'^i \partial \dot{x}'^j} = \frac{\partial}{\partial \dot{x}'^i} \left(\frac{\partial \psi}{\partial \dot{x}^k} \frac{\partial x^k}{\partial x'^j} \right) = \frac{\partial^2 \psi}{\partial \dot{x}^l \partial \dot{x}^k} \frac{\partial x^l}{\partial x'^i} \frac{\partial x^k}{\partial x'^j}.$$

Thus $\partial^2 \psi / \partial \dot{x}^i \partial \dot{x}^j$ is a covariant tensor of the second rank. Similarly $\partial^3 \psi / \partial \dot{x}^i \partial \dot{x}^j \partial \dot{x}^k$ is a covariant tensor of the third rank.

We shall call

$$f_{ij} = \frac{1}{2} \frac{\partial^2}{\partial \dot{x}^i \partial \dot{x}^j} F(x; \dot{x})$$

the *fundamental tensor*, noting that if (1.3) is true, $f_{ij} = g_{ij}$. Since F is homogeneous of the second degree in the derivatives of the coördinates, f_{ij} is homogeneous of zero degree. Therefore Euler's theorem gives

$$\frac{\partial f_{ij}}{d \dot{x}^k} \dot{x}^k = 0. \tag{2.4}$$

Also, obviously,

$$\frac{\partial f_{ij}}{\partial \dot{x}^k} \dot{x}^j = 0. \tag{2.5}$$

Using the homogeneity conditions we find

$$ds^2 = F(x; \dot{x}) dt^2 = f_{ij} dx^i dx^j, \tag{2.6}$$

a formula analogous to (1.3).

Defining f^{ij} as the minor of the determinant $f = \|f_{mn}\|$ corresponding to f_{ij}, preceded by the proper sign and divided by f, we have

$$f^{ij} f_{kj} = \delta^i_k (= 1 \text{ for } i = k; = 0 \text{ for } i \neq k) \tag{2.7}$$

and the ordinary mode of proof establishes that f^{ij} is a contravariant tensor of the second rank. (Cf. Eddington, *Report on the Relativity Theory of Gravitation*, 1920, p. 35.)

3. Defining the geodesics as curves of stationary length, we obtain from the calculus of variations the equations

$$(3.1) \qquad G_i \equiv \frac{d}{dt}\frac{\partial \sqrt{F}}{\partial \dot{x}^i} - \frac{\partial \sqrt{F}}{\partial x^i} = 0,$$

where $F = F(x; \dot{x})$, which equations retain their form for transformation of the parameter. For any other system of coördinates x'^i,

$$G'_i = \frac{d}{dt}\frac{\partial \sqrt{F}}{\partial \dot{x}'^i} - \frac{\partial \sqrt{F}}{\partial x'^i},$$

or, since $\partial \sqrt{F}/\partial \dot{x}^i$ is covariant,

$$G'_i = \frac{d}{dt}\left(\frac{\partial \sqrt{F}}{\partial \dot{x}^j}\frac{\partial x^j}{\partial x'^i}\right) - \frac{\partial \sqrt{F}}{\partial \dot{x}^j}\frac{\partial \dot{x}^j}{\partial x'^i} - \frac{\partial \sqrt{F}}{\partial x^j}\frac{\partial x^j}{\partial x'^i}$$

$$= G_j\frac{\partial x^j}{\partial x'^i} + \frac{\partial \sqrt{F}}{\partial \dot{x}^j}\left(\frac{d}{dt}\frac{\partial x^j}{\partial x'^i} - \frac{\partial \dot{x}^j}{\partial x'^i}\right)$$

and thus, by (2.3), G_i is a covariant vector.

An explicit form of the geodesic equations is obtained as follows:

For any curve, choose $t = s$, so that $F = 1$ along the curve. Then

$$\frac{1}{2}\frac{d}{ds}\frac{\partial F}{\partial \dot{x}^i} - \frac{1}{2}\frac{\partial F}{\partial x^i} = \frac{d\sqrt{F}}{ds}\frac{\partial \sqrt{F}}{\partial \dot{x}^i} + \sqrt{F}\left(\frac{d}{ds}\frac{\partial \sqrt{F}}{\partial \dot{x}^i} - \frac{\partial \sqrt{F}}{\partial x^i}\right)$$

and therefore

$$(3.2) \qquad G_i = \frac{1}{2}\frac{d}{ds}\frac{\partial F}{\partial \dot{x}^i} - \frac{1}{2}\frac{\partial F}{\partial x^i}.$$

But $F = f_{ij}\dot{x}^i\dot{x}^j$, and thus

$$G_i = \frac{d}{ds}\left(f_{ij}\dot{x}^j + \frac{1}{2}\frac{\partial f_{jk}}{\partial \dot{x}^i}\dot{x}^j\dot{x}^k\right) - \frac{1}{2}\frac{\partial f_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k$$

$$= \frac{d}{ds}(f_{ij}\dot{x}^j) - \frac{1}{2}\frac{\partial f_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k, \text{ by (2.5)},$$

and, again using (2.5), we obtain

$$(3.3) \qquad G_i = f_{ij}\ddot{x}^j + \begin{bmatrix} j\ k \\ i \end{bmatrix}\dot{x}^j\dot{x}^k,$$

where

$$2\begin{bmatrix} j\ k \\ i \end{bmatrix} = \frac{\partial f_{ij}}{\partial x^k} + \frac{\partial f_{ik}}{\partial x^j} - \frac{\partial f_{jk}}{\partial x^i}.$$

Hence

$$G^i = f^{il} G_l = \ddot{x}^i + \left\{ {j\;k \atop i} \right\} \dot{x}^j \dot{x}^k \tag{3.4}$$

where

$$\left\{ {j\;k \atop i} \right\} = f^{il} \left[{j\;k \atop l} \right],$$

and if the curve is a geodesic, its equations take the well known form for parameter s,

$$\ddot{x}^i + \left\{ {j\;k \atop i} \right\} \dot{x}^j \dot{x}^k = 0. \tag{3.5}$$

The Christoffel symbol is homogeneous of zero degree in the first derivatives of the coördinates with respect to s.

4. The equations of Levi-Civita for parallel propagation of a vector along a curve may easily be modified to meet the case of our more general metric. Let

$$2\overline{\left[{j\;k \atop i} \right]} = \frac{\partial f_{ij}}{\partial \dot{x}^k} + \frac{\partial f_{ik}}{\partial \dot{x}^j} - \frac{\partial f_{jk}}{\partial \dot{x}^i} = \frac{1}{2} \frac{\partial^3 F}{\partial \dot{x}^i \, \partial \dot{x}^j \, \partial \dot{x}^k},$$

$$\overline{\left\{ {j\;k \atop i} \right\}} = f^{il} \overline{\left[{j\;k \atop l} \right]}.$$

Let Y^i be defined as

$$Y^i = \dot{X}^i + \overline{\left\{ {j\;k \atop i} \right\}} X^j \ddot{x}^k + \left\{ {j\;k \atop i} \right\} X^j \dot{x}^k, \tag{4.1}$$

where X^i is a contravariant vector given as a function of t along a curve $x^i = x^i(t)$. For the coördinate system x'^i,

$$\begin{aligned} Y'^i &= \dot{X}'^i + \frac{1}{2} f'^{il} X'^j \left\{ \frac{\partial f'_{lj}}{\partial \dot{x}'^k} \ddot{x}'^k + \left(\frac{\partial f'_{lj}}{\partial x'^k} + \frac{\partial f'_{lk}}{\partial x'^j} - \frac{\partial f'_{jk}}{\partial x'^l} \right) \dot{x}'^k \right\} \\ &= \dot{X}'^i + \frac{1}{2} f'^{il} X'^j \left\{ \frac{d}{dt} (f'_{lj}) + \left(\frac{\partial f'_{lk}}{\partial x'^j} - \frac{\partial f'_{jk}}{\partial x'^l} \right) \dot{x}'^k \right\} \\ &= \frac{d}{dt} \left(X^m \frac{\partial x'^i}{\partial x^m} \right) + \frac{1}{2} f'^{il} X'^j \left\{ \frac{d}{dt} \left(f_{mn} \frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^l} \right) \right. \\ &\qquad \left. + \left[\frac{\partial}{\partial x'^j} \left(f_{mn} \frac{\partial x^m}{\partial x'^l} \frac{\partial x^n}{\partial x'^k} \right) - \frac{\partial}{\partial x'^l} \left(f_{mn} \frac{\partial x^m}{\partial x'^j} \frac{\partial x^n}{\partial x'^k} \right) \right] \dot{x}'^k \right\}. \end{aligned}$$

Thus

$$Y'^i - Y^m \frac{\partial x'^i}{\partial x^m} = X^m \frac{d}{dt}\left(\frac{\partial x'^i}{\partial x^m}\right) + \frac{1}{2} f'^{il} X'^j \Big\{ f_{mn} \frac{d}{dt}\left(\frac{\partial x^m}{\partial x'^j}\,\frac{\partial x^n}{\partial x'^l}\right)$$

$$+ f_{mn}\left[\frac{\partial}{\partial x'^j}\left(\frac{\partial x^m}{\partial x'^l}\,\frac{\partial x^n}{\partial x'^k}\right) - \frac{\partial}{\partial x'^l}\left(\frac{\partial x^m}{\partial x'^j}\,\frac{\partial x^n}{\partial x'^k}\right)\right]\dot{x}'^k$$

$$+ \frac{\partial f_{mn}}{\partial \dot{x}^p}\left[\frac{\partial \dot{x}^p}{\partial x'^j}\,\frac{\partial x^m}{\partial x'^l}\,\frac{\partial x^n}{\partial x'^k} - \frac{\partial \dot{x}^p}{\partial x'^l}\,\frac{\partial x^m}{\partial x'^j}\,\frac{\partial x^n}{\partial x'^k}\right]\dot{x}'^k \Big\}.$$

But, by (2.5),

$$\frac{\partial f_{mn}}{\partial \dot{x}^p}\,\frac{\partial x^n}{\partial x'^k}\,\dot{x}'^k = \frac{\partial f_{mn}}{\partial \dot{x}^p}\,\dot{x}^n = 0;$$

hence, using (2.3),

$$\begin{aligned} Y'^i - Y^m \frac{\partial x'^i}{\partial x^m} &= X^m \frac{\partial \dot{x}'^i}{\partial x^m} + f'^{il} X'^j f_{mn} \frac{\partial \dot{x}^m}{\partial x'^j}\,\frac{\partial x^n}{\partial x'^l} \\ &= X^m \frac{\partial \dot{x}'^i}{\partial x^m} + f'^{il} X^p \frac{\partial x'^j}{\partial x^p} f'_{qr} \frac{\partial x'^q}{\partial x^m}\,\frac{\partial x'^r}{\partial x^n}\,\frac{\partial \dot{x}^m}{\partial x'^j}\,\frac{\partial x^n}{\partial x'^l} \\ &= X^p \left(\frac{\partial \dot{x}'^i}{\partial x^p} + \frac{\partial x'^j}{\partial x^p}\,\frac{\partial x'^i}{\partial x^m}\,\frac{\partial \dot{x}^m}{\partial x'^j}\right) \\ &= X^p \left(\frac{\partial \dot{x}'^i}{\partial x^p} - \frac{\partial x'^i}{\partial x^m}\,\frac{\partial \dot{x}^m}{\partial \dot{x}'^j}\,\frac{\partial \dot{x}'^j}{\partial x^p}\right) \\ &= 0, \text{ by (2.2).} \end{aligned}$$

Thus Y^i is a contravariant vector, and we shall define parallel propagation of X^i by the equations

$$Y^i \equiv \dot{X}^i + \overline{\begin{Bmatrix} j\;k \\ i \end{Bmatrix}}\, X^j \ddot{x}^k + \begin{Bmatrix} j\;k \\ i \end{Bmatrix} X^j \dot{x}^k = 0, \tag{4.2}$$

which reduce to the equations of Levi-Civita when (1.3) is true.

5. We shall now proceed to the definition of *angle*. Let A be any point and p, q two curves emanating from A. Let P, Q be points on p, q respectively, such that the arcs AP, AQ are each equal to s. Let $PQ = \sigma$. We shall define the angle θ between p and q by the equation

$$\cos\theta = 1 - \frac{1}{2} \lim_{s \to 0} \frac{\sigma^2}{s^2}. \tag{5.1}$$

If the coördinates of A are x^i and those of P, Q are $x^i + dx^i$, $x^i + \delta x^i$ respectively, we find

$$\cos\theta = 1 - \frac{1}{2} F\left(x;\ \frac{dx}{\sqrt{F(x;\,dx)}} - \frac{\delta x}{\sqrt{F(x;\,\delta x)}}\right). \tag{5.2}$$

But the expression on the right retains the same value if $dx^1, \cdots, dx^N$ or $\delta x^1, \cdots, \delta x^N$ are replaced by quantities proportional to them, and therefore (5.2) defines the angle between the curves if $dx, \delta x$ are any infinitesimal displacements in the directions of the curves. Since F is homogeneous of the second degree in its directional arguments, the angle thus defined does not depend on the order in which the curves are considered.

There is, however, another definition of angle which extends the fundamental property of parallel propagation in Riemannian space into the more general type under consideration. If we are given two vectors X^i, Y^i at a point on a curve C, we define *the angle* $\Phi(X, Y; C)$ *between the vectors, with respect to* C, by

$$\cos \Phi(X, Y; C) = \frac{f_{ij} X^i Y^j}{\sqrt{f_{mn} X^m X^n \cdot f_{pq} Y^p Y^q}}, \tag{5.3}$$

where the directional arguments of the f's are the coördinate derivatives $\dot{x}^i$ along C. Now if X and Y undergo parallel propagation along C,

$$\begin{aligned} \frac{d}{dt}(f_{ij} X^i Y^j) &= -f_{ij}\left(\left\{\begin{matrix}\overline{l\ k}\\ i\end{matrix}\right\}\ddot{x}^k + \left\{\begin{matrix}l\ k\\ i\end{matrix}\right\}\dot{x}^k\right)(X^l Y^j + X^j Y^l) \\ &\qquad + \left(\frac{\partial f_{ij}}{\partial \dot{x}^k}\ddot{x}^k + \frac{\partial f_{ij}}{\partial x^k}\dot{x}^k\right) X^i Y^j \\ &= X^i Y^j\left[\left(\frac{\partial f_{ij}}{\partial \dot{x}^k} - \left[\begin{matrix}\overline{i\ k}\\ j\end{matrix}\right] - \left[\begin{matrix}\overline{j\ k}\\ i\end{matrix}\right]\right)\ddot{x}^k + \left(\frac{\partial f_{ij}}{\partial x^k} - \left[\begin{matrix}i\ k\\ j\end{matrix}\right] - \left[\begin{matrix}j\ k\\ i\end{matrix}\right]\right)\dot{x}^k\right] \\ &= 0. \end{aligned}$$

Similarly the denominator in (5.3) also has a zero derivative, and thus *the angle between two vectors, with respect to a curve, remains constant when both vectors undergo parallel propagation along the curve.*

The foregoing definition of angle with respect to a curve gives us at once a definition of *perpendicularity of* Y *with respect to* X, expressed by the relation

$$f_{ij} X^i Y^j = \frac{1}{2}\frac{\partial F(x; X)}{\partial X^j} Y^j = 0, \tag{5.4}$$

where the directional arguments of f_{ij} are the components of X. We say, then, that *two vectors are perpendicular with respect to one another* if

$$\frac{\partial F(x; X)}{\partial X^i} Y^i = \frac{\partial F(x; Y)}{\partial Y^i} X^i = 0. \tag{5.5}$$

This last idea leads to consideration of a type of principal direction in a two-dimensional space, non-existent for the Riemannian metric; *those directions may be termed principal which are perpendicular with respect to one another.*

As a simple illustration, let

$$ds = \sqrt[4]{dx^{1^4} + dx^{2^4}}\,. \tag{5.6}$$

Then the conditions (5.5) give

$$X^{1^3}Y^1 + X^{2^3}Y^2 = Y^{1^3}X^1 + Y^{2^3}X^2 = 0.$$

Therefore

$$Y^1 = \pm Y^2,$$

$$X^1 = \mp X^2,$$

and the differential equations of the principal directions are

$$dx^1 \pm dx^2 = 0. \tag{5.7}$$

As in Riemannian geometry, every null-direction is perpendicular to itself in the sense of (5.4).

6. In the case where $ds^2 = F(x;\, dx)$ is a function of the differentials only, as in (5.6), f_{ij} are independent of the coördinates x^i, and $\left\{ {jk \atop i} \right\} = 0$. Thus, by (3.5), the equations of the geodesics are

$$\frac{d^2x^i}{ds^2} = 0, \tag{6.1}$$

whence

$$x^i = \lambda^i s + \alpha^i. \tag{6.2}$$

Therefore for such types of line-element, the axioms of connection and order hold, as well as those axioms of congruence which do not deal with angles. Planes exist and the euclidean axiom of parallels is true.

For parallel propagation along any geodesic, we find from (4.2) and (6.1)

$$X^i = \text{constant}. \tag{6.3}$$

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